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FREE THERMOELASTIC SHELLS*

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Two-dimensional equations for free thermoelastic shells are obtained by an asymptotic method /1/ from the three-dimensional equations of thermoelasticity. It hence turns out that the Kirchhoff hypothesis on invariability of the length of the normal for a nonlinear law of the temperature variation over the thickness can result in substantial errors, in which connection new terms are introduced in the elasticity relations.

Separation of the internal state of stress into a main state of stress and simple edge effects is often used to analyze unheated shells subjected to the effect of an external load. The method of separation is extended here to free thermoelastic shells.

1. Even in the roughest approximation the equilibrium equations for free shells allow no simplifications, hence we shall construct only the elasticity relationships.

The equations and notation used here are taken from /1,2/.

Let lines α_i coincide with the lines of curvature on the shell middle surface, and let the γ-lines be orthogonal.

We take the three-dimensional equations of state of thermoelasticity as the initial equations

$$\tau_{i*} = \frac{E}{R(1-\nu^2)}\left(\frac{a_j}{a_i} e_{i*} + \nu e_{j*}\right) + \eta^1 \frac{\nu}{1-\nu} \frac{1}{a_i} \tau_{3*} - \frac{a_j}{1-\nu} T_*, \quad \eta^{1-2s}\tau_{ij*} = \frac{E}{2R(1+\nu)}\left(\frac{a_i}{a_j} m_{i*} + m_{j*}\right) \tag{1.1}$$

$$\frac{\partial U_{3*}}{\partial \zeta} = \eta^2 \frac{R}{E a_1 a_2} \tau_{3*} - \eta^1 \nu \frac{1}{E} R\left(\frac{\tau_{1*}}{a_2} + \frac{\tau_{2*}}{a_1}\right) + \eta_*^{\,1} \frac{R}{E} T_*, \quad \frac{\partial v_{i*}}{\partial \zeta} = -\eta^{1-2s} \frac{g_{i*}}{a_i} + \eta^{2-2s} 2(1+\nu) \frac{R}{E a_j} \tau_{i3*}$$

$$a_i = 1 + \eta\zeta \frac{R}{R_i} \quad (i \neq j = 1, 2)$$

(E is the elastic modulus, ν is the Poisson's ratio, τ_i, τ_{ij}, τ_3, τ_{i3} are stresses, v_i, v_3 are displacements, R_i are radii of curvature, and R is some characteristic dimension). In place of the symmetric stress tensor here, a nonsymmetric stress tensor τ_i, τ_{ij}, τ_{i3}, τ_3 is introduced, but since the stresses of both tensors differ by the quantities $O(\eta^1)$, (η is a small quantity equal to the ratio between half the shell thickness h and the characteristic dimension R), then this difference can be neglected to the accuracy of the quantities $O(\eta^1)$.

The e_i, m_i, g_i in (1.1) are expressed in terms of the displacements v_i, v_3 as follows

$$e_{i*} = \frac{1}{A_i} \frac{\partial v_{i*}}{\partial \xi_i} + \eta^s R k_i v_{j*} + \frac{R}{R_i} v_{3*}, \quad m_{i*} = \frac{1}{A_j} \frac{\partial v_{i*}}{\partial \xi_j} - \eta^s R k_j v_{j*} \tag{1.2}$$

$$g_{i*} = \frac{1}{A_i} \frac{\partial v_{3*}}{\partial \xi_i} - \eta^{2-2s} \frac{R}{R_i} v_{i*}, \quad k_i = \frac{1}{A_i A_j} \frac{\partial A_i}{\partial \alpha_j} \ (i \neq j = 1,2)$$

The quantities R, η, ξ_i, ζ, T_* in the preceding formulas should be replaced by $1, 1, \alpha_i, \gamma$, $E\alpha_t T$, respectively, and the asterisks on the desired quantities should be omitted. This notation is needed later.

Let us write down the condition on the face surfaces of the shell (there is no surface load)

$$\tau_3 |_{\gamma=\pm h} = 0, \quad \tau_{i3} |_{\gamma=\pm h} = 0 \tag{1.3}$$

2. We shall consider the temperature field known. We introduce the product of the appropriate quantity with an asterisk by η^a, instead of the temperature T, the stresses, and displacements, by selecting the exponent a in such a way that all the quantities with the asterisks would be of the same order

$$E\alpha_t T = \eta^0 T_*, \quad E v_3 = \eta^0 v_{3*}, \quad E v_i = \eta^s v_{i*}, \quad \tau_{ij} = \eta^{1-2s} \tau_{ij*}, \quad \tau_i = \eta^0 \tau_{i*} \tag{2.1}$$

$$O\left(\int_{-1}^{+1} \tau_i \, d\zeta, \quad \int_{-1}^{+1} \tau_{ij} \, d\zeta\right) = O(\eta^{2-2s}\tau_i), \quad O\left(\int_{-1}^{+1} \tau_i \zeta \, d\zeta\right) = O(\eta^{1-2s}\tau_i), \quad \tau_{i3} = \eta^{1-s} \tau_{i3*}, \quad \tau_3 = h^1 \tau_{3*}$$

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Here s is the index of variability of the principal state of stress. The asymptotic representation (2.1) has been selected by using reasoning analogous to that in /1/. The powers of η there were selected in such a way that the boundary value problems that are obtained for the asymptotic taken as $\eta \to 0$ would be consistent.

Let us execute the scale stretching in the coordinates α_i, γ customary for the asymptotic method

$$\frac{\partial}{\partial \alpha_i} = \eta^{-s} \frac{1}{R} \frac{\partial}{\partial \xi_i}, \quad \frac{\partial}{\partial \gamma} = \eta^{-1} \frac{1}{R} \frac{\partial}{\partial \zeta} \tag{2.2}$$

Differentiation of the required functions with respect to the varialbes ξ_i, ζ introduced in such manner does not result in a substantial increase in that functions.

Substituting (2.1) and (2.2) into the three-dimensional thermoelasticity equations, we obtain (1.1) in which E should be considered equal to unity.

For convenience in the subsequent calculations, we represent the stresses τ_{i*}, τ_{3*} in the form of sums

$$\tau_{i*} = \eta^{1-2s}\tau_{i*}' + \tau_{i*}'', \quad \tau_{3*} = \eta^{1-2s}\tau_{3*}' + \tau_{3*}'' \tag{2.3}$$

Here

$$\eta^{1-2s}\tau_i' = \frac{1}{R(1-\nu^2)}\left(\frac{a_j}{a_i} e_{i*} + \nu e_{j*}\right) + \eta \frac{\nu}{1-\nu} \frac{\tau_{3*}}{a_i} + \frac{1-a_j}{1-\nu} T_* -$$
$$\frac{1}{2(1-\nu)} \int_{-1}^{+1} T_* d\zeta, \quad \tau_i'' = \frac{1}{2(1-\nu)} \int_{-1}^{+1} T_* d\zeta - \frac{1}{1-\nu} T_* \tag{2.4}$$

The component τ_i'' has been selected so that the force calculated by using it would be zero, and we determine τ_3'' by integrating the third equilibrium equation after having substituted τ_i''.

We use $O(\eta^1)$ accuracy in defining the stresses τ_i, τ_{ij}. Then as follows from (1.1) and (2.4), the displacement and temperature should be determined to the accuracy of $O(\eta^{2-2s})$.

The quantity τ_{3*} enters the first formula in (1.1) with a factor η^1 which means according to the above that we shall evaluate τ_{3*} to the accuracy η^{1-2s} in this relationship within the framework of the accuracy used. To this end, we use the third equilibrium equation, which takes the following form after small terms have been discarded /1/:

$$-\frac{\tau_{1*}''}{R_1} - \frac{\tau_{2*}''}{R_2} + \frac{\partial \tau_{3*}''}{\partial \zeta} = 0$$

Integrating this equation with respect to ζ and taking (2.4) into account, we obtain the following expression for τ_3'':

$$\tau_{3*}'' = \tau_{3,0}'' + \frac{2}{2(1-\nu)}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left[\zeta \int_{-1}^{1} T_* d\zeta - 2\int_{0}^{\zeta} T_* d\zeta\right] \tag{2.5}$$

We find the displacements by integrating the last two equations of (1.1) with respect to ζ while taking account of (2.3) and (2.4)

$$v_{i*} = v_{i,0} - \eta^{1-2s}\zeta g_{i,0}, \quad v_{3*} = v_{3,0} + \eta^1 \frac{1+\nu}{1-\nu} R \int_0^{\zeta} T_* d\zeta - \eta^1 \frac{\nu}{1-\nu} R\zeta \int_{-1}^{+1} T_* d\zeta \tag{2.6}$$

The quantities with zero subscript are independent of ζ.

We substitute (2.6) into (1.2), and consequently obtain

$$e_{i*} = e_{i,0} - \eta^{1-2s}\zeta\left(\frac{1}{A_i}\frac{\partial g_{i,0}}{\partial \xi_i} + \eta^s R k_i g_{j,0}\right) + \eta^1 \frac{R^2}{R_i}\left(\frac{1+\nu}{1-\nu}\int_0^{\zeta} T_* d\zeta - \frac{\nu}{1-\nu}\zeta \int_{-1}^{+1} T_* d\zeta\right) \tag{2.7}$$

$$m_{i*} = m_{i,0} - \eta^{1-2s}\zeta\left(\frac{1}{A_j}\frac{\partial g_{i,0}}{\partial \xi_j} - \eta^s R k_j g_{j,0}\right)$$

Here $g_{i,0}, e_{i,0}, m_{i,0}$ are interpreted by means of (1.2) in which the asterisks must be replaced by zeroes.

Let us go over to the notation used in shell theory in the formulas obtained. The displacements u_i, w of the middle surface, the strains, the forces, and the moments are expressed in terms of the desired quantities of three-dimensional theory as follows:

$$\frac{1}{E} v_{i,0} = \eta^{-s} u_i, \quad \frac{1}{E} v_{3,0} = -w, \quad \frac{1}{E} e_{i,0} = R\varepsilon_i, \quad \frac{1}{E} g_{i,0} = \eta^s R\gamma_i, \quad \frac{1}{E} m_{i,0} = R\omega_j \tag{2.8}$$

$$T_i = \int_{-h}^{+h} \tau_i \, d\gamma, \quad G_i = -\int_{-h}^{+h} \tau_i \gamma \, d\gamma, \quad S_{ij} = \int_{-h}^{+h} \tau_{ij} \, d\gamma, \quad H_{ij} = \int_{-h}^{+h} \tau_{ij} \gamma \, d\gamma$$

The strains $\varepsilon_i, \gamma_i, \omega_j$ are determined in terms of the displacements of the middles surface by the formulas presented in /1/.

Taking (1.1), and (2.3)—(2.8) into account, we obtain the following elasticity relationships for the moments after simple, but awkward, computations:

$$G_i = -\frac{2Eh^3}{3(1-\nu^2)}(\varkappa_i + \nu\varkappa_j) + \frac{E\alpha_t}{1-\nu}\left\{\frac{h^2}{3}\left(\frac{1}{R_i} - \frac{1}{R_j}\right)\int_{-h}^{+h} T\,d\gamma - \frac{1}{R_i}\int_{-h}^{h}\gamma\left(\int_0^{\gamma} T\,d\gamma\right)d\gamma + \frac{1}{R_j}\int_{-h}^{+h} T\gamma^2\,d\gamma\right\} + \frac{E\alpha_t}{1-\nu}\int_{-h}^{+h} T\gamma\,d\gamma, \quad H_{ij} = \frac{2Eh^3}{3(1+\nu)}\tau \tag{2.9}$$

The terms in the braces equal zero, when T is a linear function in γ.

In place of the elasticity relationships for the normal and shear forces, we obtain

$$\varepsilon_i = \frac{\alpha_t}{2h}\int_{-h}^{+h} T\,d\gamma, \quad \omega = 0 \tag{2.10}$$

For a linear law of temperature variation over the thickness $T = t_0 + \gamma t_1$ the elasticity relationship for G_i takes the usual form

$$G_i = -\frac{2Eh^3}{3(1-\nu^2)}(\varkappa_i + \nu\varkappa_j) + \frac{2Eh^3\alpha_t}{3(1-\nu)}t_1 \tag{2.11}$$

3. Let us separate the boundary conditions on the free edge into boundary conditions for the principal state of stress and for the simple edge effect.

Following /3/, we represent each of the quantities of the stress-strain state (displacements, forces, moments) in the form of a sum

$$P = P^{(g)} + \eta^c P^{(eg)} \tag{3.1}$$

The superscripts (g), (eg) show that this quantity belongs to the principal state of stress or the simple edge effects, respectively. The quantities $P^{(g)}$ are found from the inhomogeneous equations (2.9), (2.10) and the equilibrium equations, while the quantities $P^{(eg)}$ are found from the homogeneous equations of the simple edge effect, hence the scale factors η^c are theirs, where the number c will be selected as a function of the boundary conditions.

Let us write the asymptotic representation for the forces and moments of the principal state of stress (2.1)

$$T_1^{(g)} = T_{1*}^{(g)}, \quad S_{12}^{(g)} = S_{12*}^{(g)}, \quad G_1^{(g)} = \eta^{2s}G_{1*}^{(g)}, \quad N_1^{(g)} = \eta^s N_{1*}^{(g)} \tag{3.2}$$

and of the simple edge effect

$$T_1^{(eg)} = T_{1*}^{(eg)}, \quad S_{12}^{(eg)} = \eta^{-s}S_{12*}^{(eg)}, \quad G_1^{(eg)} = \eta^{1/2}G_{1*}^{(eg)}, \quad N_1^{(eg)} = N_{1*}^{(eg)} \tag{3.3}$$

in the boundary conditions.

The boundary conditions on the free edge $\alpha_1 = \alpha_{10}$ have the form

$$T_1' = 0, \quad S_{12}' = 0, \quad G_1 = 0, \quad N_1' = 0 \tag{3.4}$$

Here T_1', S_{12}', N_1' are the reduced edge forces. By using (3.1)—(3.3), they can be represented as follows

$$T_{1*}^{'(g)} + \eta^c T_{1*}^{'(eg)} = 0, \quad S_{12*}^{'(g)} + \eta^{c-s}S_{12*}^{'(eg)} = 0, \quad \eta^{2s}G_{1*}^{(g)} + \eta^{c+1/2}G_{1*}^{(eg)} = 0, \quad \eta^s N_1^{'(g)} + \eta^c N_{1*}^{'(eg)} = 0 \tag{3.5}$$

We neglect quantities on the order of ε where

$$\varepsilon = O(\eta^{1/2-s}) \tag{3.6}$$

when separating the boundary conditions.

To the accuracy taken, the primes can be omitted on the quantities of the edge effect in (3.5). We take the following number as c

$$c = -1/2 + 2s \tag{3.7}$$

In the roughest approximation for the simple edge effect we obtain the boundary conditions

$$G_1^{(eg)} = -G_1^{(g)}, \quad N_1^{(eg)} = 0 \tag{3.8}$$

from (3.5) with (3.6) and (3.7) taken into account.

We express $T_1^{(eg)}$ and $S_{12}^{(eg)}$ on the edge in terms of the edge values of the forces and moments of the principal state of stress. To this end we use formulas for the magnitudes of the simple edge effect which are presented in /3/.

Let us represent each of the quantities of the simple edge effect in the form of the expansions

$$P_*^{(eg)} = \sum_{i=0}^{\infty} \eta^{i(1/2-s)} P_{,i}^{(eg)} \tag{3.9}$$

The number i after the comma denotes the number of the approximation.

To determine $T_{1,0}^{(eg)}$ and $S_{12,0}^{(eg)}$ at the edge, we take formulas for the zero-th approximation in /3/ and satisfy the boundary conditions (3.8), whereupon we obtain

$$T_{1,0}^{(eg)} = S_{12,0}^{(eg)} = 0 \quad (\alpha_1 = \alpha_{10}) \tag{3.10}$$

We insert the expansions of the forces $T_1^{(eg)}$ and $S_{12}^{(eg)}$ of the form (3.9) into (3.5), we take account of (3.7), and (3.10), and we consequently obtain to the accuracy of (3.6)

$$T_{1*}^{\prime(g)} + T_{1,1}^{(eg)} = 0, \quad S_{12*}^{\prime(g)} + S_{12,1}^{(eg)} = 0 \tag{3.11}$$

$$G_{1,1}^{(eg)} = -G_{1**}^{(g)}, \quad N_{1,1}^{(eg)} = -N_{1*}^{\prime(g)} \tag{3.12}$$

The new quantity $G_{1**}^{(g)}$ is introduced into (3.12). It is $\eta^{-1/2+s}$ times less than $G_{1*}^{(g)}$ and is determined in the computation of the next, more exact approximation of the principal state of stress. As will be shown below, $G_{1**}^{(g)}$ will not enter into the final result.

We find the values of the forces $T_{1,1}^{(eg)}$ and $S_{12,1}^{(eg)}$ at the edge by using formulas of the first-approximation of the edge effect from /3/ and the boundary conditions (3.12). We then substitute the values obtained $T_{1,1}^{(eg)}$ and $S_{12,1}^{(eg)}$ into (3.11), and the boundary conditions for the principal state of stress will then become

$$\begin{gathered} T_1^{\prime(g)} + A_1\left(\frac{1}{A_2}\frac{\partial}{\partial\alpha_2} + 2k_1\right)\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\frac{R_2}{A_1}G_1^{(g)} - \frac{1}{R_1}(k_2^2R_2^2+1)\,G_1^{(g)} + k_2R_2N_1^{\prime(g)} = 0 \\ S_{12}^{\prime(g)} - \left[\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}k_2\left(1-\frac{R_2}{R_1}\right) - 2k_2\frac{1}{A_2}\frac{\partial}{\partial\alpha_2} + k_1k_2\right]R_2G_1^{(g)} - \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}R_2N_1^{\prime(g)} = 0 \end{gathered} \tag{3.13}$$

It should be noted that conditions (3.13) and (3.8) are suitable for a computation of unheated free shells by the separation method since what caused the state of stress, the temperature field or the external load, did not play any part in their construction.

We show by an example that the new terms introduced into the elasticity relationship (2.9) can alter the results of the computation substantially. Let us consider a free circular cylindrical shell of radius r and length πr with quadratic temperature $T = t_0\gamma^2 \sin\xi$ over the thickness (ξ is the coordinate along the generatrix, $0 \leq \xi \leq \pi$).

Using (2.10) and (2.11), we obtain the following values for the bending moments:

$$G_1 = -\frac{2}{9}\frac{Eh^5t_0\alpha_t}{r(1-\nu^2)}\left[1 - \frac{4(1+\nu)^\circ}{5}\right], \quad G_2 = -\frac{2}{9}\frac{Eh^5t_0\alpha_t}{r(1-\nu^2)} \times \left[\nu - \frac{2(1+\nu)^\circ}{5}\right]$$

The terms marked with the degree symbol appeared becuase of taking account of elongation of the normal element. They are missing if any other theory of thermoelastic shells is used.

Let us estimate the error of the constructed theory. The total error is comprised of the error in the equations, equal to η^1, and the error in the boundary conditions (3.6), and since the total error equals the greatest of those admitted, then it is determined by (3.6).

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